ON THE ADDITIVITY OF THE THURSTON–BENNEQUIN INVARIANT OF LEGENDRIAN KNOTS

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Abstract. In this article, we consider the maximal value of the Thurston–Bennequin invariant of Legendrian knots which topologically represent a fixed knot type in the standard contact 3-space and we prove a formula of the value under the connected sum operation of knots.

1. Introduction

The *standard contact structure* ξ_0 on 3-space $\mathbb{R}^3 = \{(x, y, z)\}$ is the plane field on $\mathbb{R}^3$ given by the kernel of the 1-form $dz - ydx$. A *Legendrian knot* K in the contact manifold $(\mathbb{R}^3, \xi_0)$ is a knot which is everywhere tangent to the contact structure ξ_0. The *Thurston–Bennequin invariant* $tb(K)$ of a Legendrian knot K in $(\mathbb{R}^3, \xi_0)$ is the linking number of K and a knot K' which is obtained by moving K slightly along the vector field $\frac{\partial}{\partial z}$. For a topological knot type k in $\mathbb{R}^3$, the *maximal Thurston–Bennequin invariant* $mtb(k)$ is defined to be the maximal value of $tb(K)$, where K is a Legendrian knot which topologically represents k. For any k, by the *Bennequin's inequality* in [1], we know that $mtb(k)$ is an integer (i.e. not ∞). There are several computations of $mtb(k)$ (for example, see [3], [5], [8], [9], [10]).

In this paper, we prove the following theorem.

Theorem 1.1. *Let $k_1 \sharp k_2$ be the connected sum of topological knots k_1 and k_2 in $\mathbb{R}^3$. Then $mtb(k_1 \sharp k_2) = mtb(k_1) + mtb(k_2) + 1$.*

Remark 1.2. After writing this paper, the author was informed that J. Etnyre and K. Honda [4] have also obtained a result on connected sum of Legendrian knots which extensively includes Theorem 1.1.

Date: February 26, 2001.

1991 *Mathematics Subject Classification.* Primary 57R17; Secondary 57M25.

Key words and phrases. Legendrian knot, Thurston–Bennequin invariant, connencted sum.

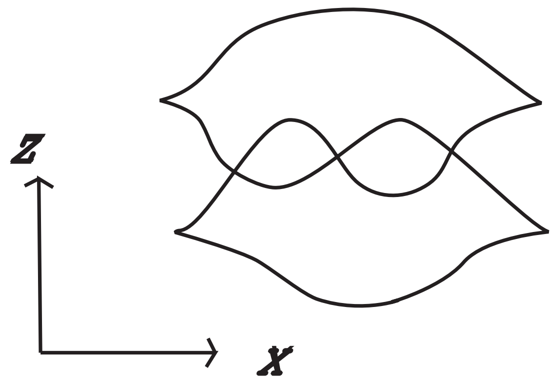

Figure 1.

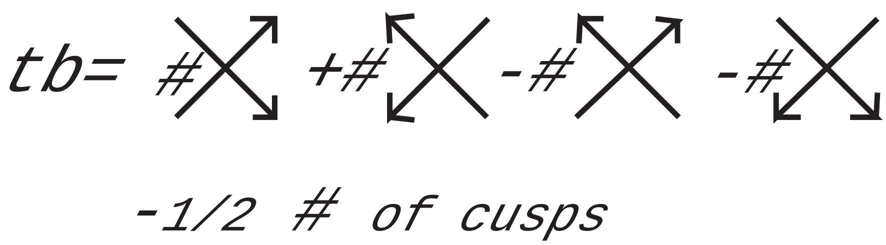

Figure 2.

2. Fronts

Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_0 = ker(dz - ydx))$. Then a diagram (i.e. projection) of K in xz-plane is called *front* as in Figure 1.

A front does not have vertical tangents; generically, its only singularities are transverse double points and semicubical cusps. Note that the number of the cusps is even. Since $y = \frac{\partial z}{\partial x}$ along K, the missing y coordinate is the slope of the front. Therefore the front of K is free from selftangencies, and, at a double point, the branch with a greater slope is higher along the y axis. Conversely such a diagram uniquely determines K as its front. So, as usual in knot theory, we identify a Legendrian knot K with its front, also denoted by K.

The Thurston–Bennequin invariant $tb(K)$ is computed in terms of the double points and cusps of its front. See Figure 2, where K is oriented and the choice of the orientaion is irrelevant for the value of $tb(K)$.

For example, $tb(K) = -5$ for the front in Figure 1.

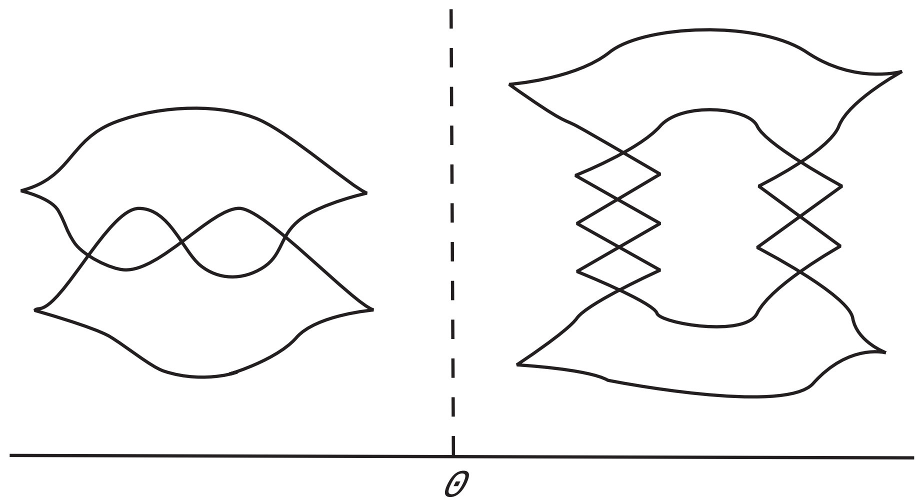

FIGURE 3.

Proposition 2.1. *For two topological knots k_1 and k_2, we have $mtb(k_1 \sharp k_2) \geq mtb(k_1) + mtb(k_2) + 1$.*

Proof. Let K_1 and K_2 be Legendrian knots whose topological types are k_1 and k_2, respectively and $mtb(k_1) = tb(K_1)$ and $mtb(k_2) = tb(K_2)$. We also regard K_1 and K_2 as fronts. Further we can assume that $K_1 \cap K_2 = \emptyset$ and K_1 (resp. K_2) lies in the left (resp. right) region of xz-plane, i.e. $\{(x,z)|x < 0\}$ (resp. $\{(x,z)|x > 0\}$) as in Figure 3.

Then we connect K_1 and K_2 by joining a right cusp of K_1 and a left cusp of K_2 as in Figure 4.

This procedure produces a Legendrian knot whose topological type is $k_1 \sharp k_2$ and Thurston–Bennequin invariant is $mtb(k_1) + mtb(k_2) + 1$.

This completes the proof.

□

3. Preliminaries from contact topology

In this section, we recall some basic notions and theorems from recent 3-dimensional contact topology. Further, we may assume the reader is familiar with convex surface theory started by E. Giroux in [6]. For details and proofs, see [2], [3], [6], [7], [8]. Let $\xi_n = ker(sin(2\pi nz)dx + cos(2\pi nz)dy)$ be the contact structure on a solid torus $V = \{(x,y,z) \in \mathbb{R}^3_z | x^2 + y^2 \leq \epsilon\}$, where $n \in \mathbb{Z} - \{0\}$ and $\mathbb{R}^3_z$ is $\mathbb{R}^3$ modulo $z \mapsto z+1$. The *characteristic foliation* on an embedded surface in a contact 3-manifold is the singular foliation defined by the intersection of the contact structure and the surface. The set of tangents of ξ_n to ∂V forms

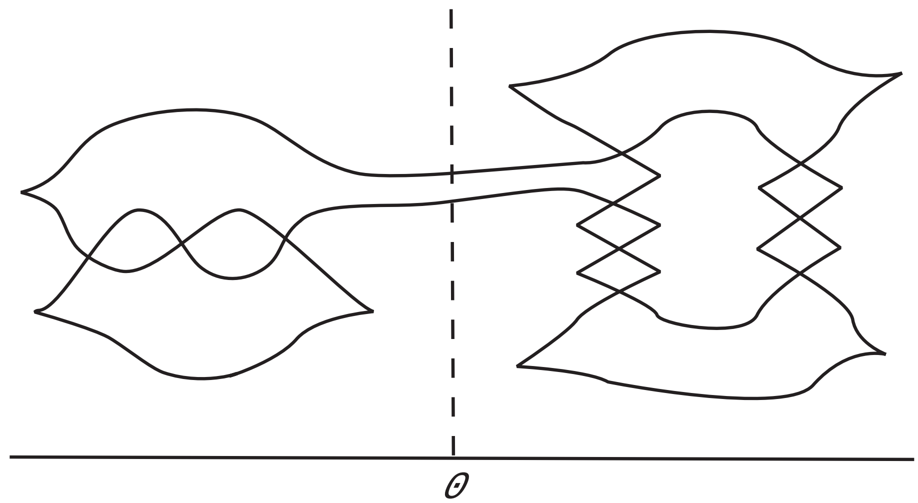

FIGURE 4.

a disjoint union of two simple closed curves on ∂V, which are called *Legendrian divides*. Legendrian divides are leaves of the characteristic foliation on ∂V.

The next lemma is proved by a standard Darboux-type argument.

Lemma 3.1. *For any Legendrian knot K in $(\mathbb{R}^3, \xi_0)$, there exists a sufficiently small neighborhood $N(K)$ such that $(N(K), K, \xi_0)$ is isomorphic to $(V, \{(0,0,z)\}, \xi_n)$ for some n.*

Note that in Lemma 3.1, if K is topologically trivial, then $n = tb(K)$.

As ∂V is a *convex surface* (i.e. has a contact vector field transverse to ∂V), the following lemma can be proved by convex surface theory.

Lemma 3.2. *Let T be any embedded torus in $(\mathbb{R}^3, \xi_0)$ and W a solid torus bounded by T. Suppose the characteristic foliation on T is diffeomorphic to that on ∂V and identifying these, the Legendrian divides on T are isotopic to the core curve of W through an isotopy in W. Then (W, ξ_0) is isomorphic to (V, ξ_n) for some n.*

The following theorem on the classification of topologically trivial Legendrian knots due to Y. Eliashberg–M. Fraser [2] is also needed for the proof of Theorem 1.1.

Theorem 3.3. *Any topologically trivial Legendrian knot is Legendrian isotopic to one of standard forms expressed as fronts in Figure 5.*

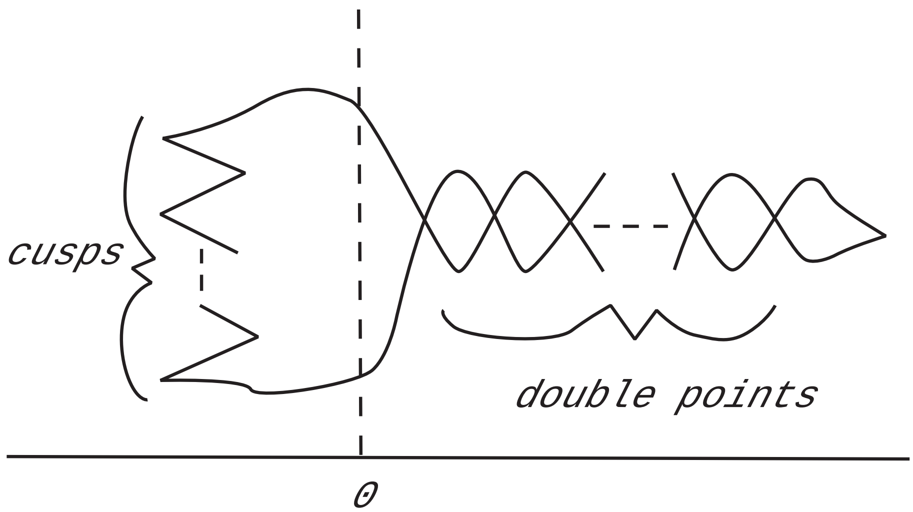

FIGURE 5.

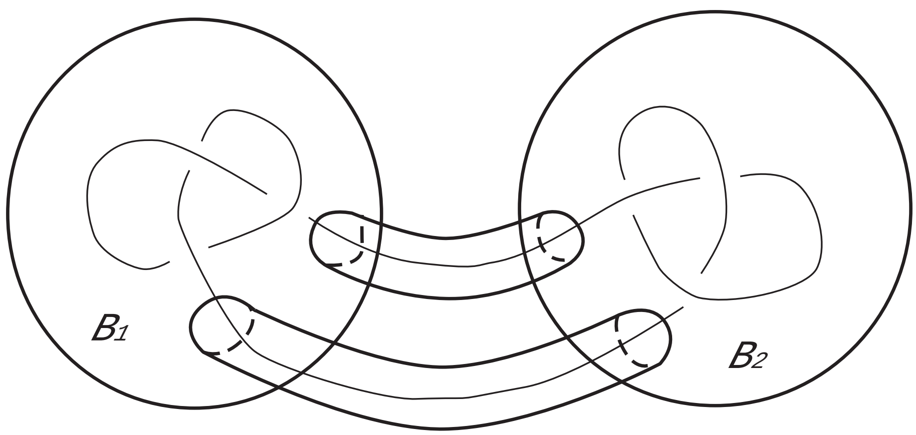

FIGURE 6.

4. Proof of Theorem 1.1

By Propositon 2.1, it is sufficient to show the converse inequality.

Suppose $\hat{K}$ is a Legendrian knot in $(\mathbb{R}^3, \xi_0)$ whose topological type is the connected sum of k_1 and k_2 and its Thurston–Bennequin invariant is maximal. By Lemma 3.1, there exists a neighbourhood $N(\hat{K})$ of $\hat{K}$ such that $(N(\hat{K}), \xi_0)$ is isomorphc to (V, ξ_n) for some n. Let B_1 and B_2 be 3-balls in $\mathbb{R}^3$ such that B_1 (resp. B_2) splits $\hat{K}$ into the component corresponding to k_1 (resp. k_2) and $B_1 \cap B_2 = \emptyset$ (Figure 6).

Further, by convex surface theory, we can assume that (i) ∂B_1 and ∂B_2 are convex and (ii) $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ are Legendrian knots on ∂B_1 and ∂B_2, respectively and (iii) each *dividing set* on ∂B_i (i.e. the subset of ∂B_i consisting of tangents of ξ_0 and a contact vector field defining the convex surface) intersects $\partial B_i \cap N(\hat{K})$ as a diameter of the disk.

Then by Edge-Rounding Lemma due to K. Honda in [7], we have a solid torus W such that (i) W equals $B_1 \cup B_2 \cup N(\hat{K})$ except small neighbourhoods of $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ and (ii) ∂W is a convex surface whose characteristic foliation is diffeomorphic to that of ∂V. By Lemma 3.2, it follows that (W, ξ_0) is isomophic to (V, ξ_n) for some n. And notice that W is unknotted in $\mathbb{R}^3$ and hence the core curve K of W which is Legendrian is also unknotted. Further, by a standard argument, we can assume that K agrees with $\hat{K}$ in the region of $N(\hat{K}) - (B_1 \cup B_2)$. So by Theorem 3.3, K is Legendrian isotopic to one of standard forms in Figure 5. Therefore W is also identified with a small neighbourhood of that of the standard form. Further, by a homogeneous property of V and a parallel translation of W, we can assume that a region of W corresponding to B_1 (resp. B_2) lies in $\{(x, y, z) | x < 0\}$ (resp. $\{(x, y, z) | x > 0\}$). Then, identifying $\hat{K}$ with its front, we can divide $\hat{K}$ along a vertical line into Legendrian knots K_1 and K_2 corresponding to k_1 and k_2, respectively as the converse procedure in the proof of Proposition 2.1.

Counting the Thurston-Bennequin invariant of K_1 and K_2, we have $tb(\hat{K}) = mtb(k_1 \sharp k_2) = tb(K_1) + tb(K_2) + 1$. Therefore $mtb(k_1 \sharp k_2) \leq mtb(k_1) + mtb(k_2) + 1$.

This completes the proof of the main theorem.

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